A Synthesis of Research on Effective Mathematics Instruction

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Effective mathematics instruction is characterized by a well integrated development of students' conceptual understanding, procedural fluency, and problem solving. As they develop these abilities, students must become facile with mathematics vocabulary and with representing mathematical ideas in multiple ways. To achieve these outcomes, effective teachers of mathematics skillfully employ a wide repertoire of strategies and approaches. The purpose of this paper is to review the research literature on effective mathematics instruction.

A. What the Research Says About Effective Instruction for Conceptual Understanding, Procedural Fluency, and Problem Solving

Conceptual understanding and procedural fluency are not "either/or" elements of mathematical knowledge – they grow together. Conceptual understanding rests on a framework of facts. Memorizing facts and skills is necessary, but not entirely sufficient for building mathematical understanding. Memorization is most effective when the facts and skills are organized in ways that allow them to be retained and recalled quickly and automatically for use in solving a new problem, confronting a new situation, or finding where in the existing schema to add or fit new information. The more facts and skills students have appropriately organized in their long-term memory schemas, the better their conceptual understanding. It is this organization of facts into conceptual frameworks that facilitates the retrieval, application, and transfer of knowledge (Bransford, Brown, & Cocking, 2000; Hiebert et al, 1997; Hirsch, 2006).

Conceptual understanding allows procedures to be appropriately selected and used flexibly. If students are taught mostly algorithms and rules based on abstract symbols (*syntactic procedures*) without opportunity to use these procedures in flexible ways to solve diverse problems, constraints are placed on problem solving ability. Problem solving entails the ability to determine what a problem is about and to form a mental picture of what the problem represents (*semantic analysis*). Syntactic procedures alone can generate correct performance on direct measures; i.e., on the tasks for which they were specifically taught. However, that correct performance does not transfer well to novel problems across time. Semantic analysis, on the other hand, does transfer because it enables students to form correct representations of new problems (Hiebert & Wearne, 1988).

Conceptual understanding involves knowing *what* to do, while procedural fluency requires knowing *how* to do it. Growth in conceptual understanding and procedural skill is a bidirectional process. Practice in using skills and procedures across a range of problems strengthens conceptual understanding, while conceptual understanding enables students to know which procedures to select for particular problems, opening the way for further practice with the procedures (Miller & Mercer, 1992; Sophian, 1997).

Explicit, systematic instruction in problem solving has been shown to benefit students of all ability levels. Teaching students both how to do it and when to do it, and offering precise, constructive feedback during guided and independent practice, *scaffolds* student learning – providing temporary supports that can eventually be removed as students gain automaticity with skills and a deeper understanding of concepts. Students learn conceptual understanding, procedural fluency, and problem solving most effectively when teachers scaffold their learning by:

- Reviewing and building on students' previous learning
- Working toward clear, explicit learning goals
- Presenting new material in manageable steps that encourage active student participation
- Modeling, explaining, and prompting
- Teaching students how to prepare and solve problems systematically
- Teaching and discussing cognitive and metacognitive strategies
- Presenting multiple examples of a concept so that students can deduce underlying principles
- Asking students to propose preliminary solutions and providing feedback as to the effectiveness of their thinking
- Providing regular practice with ongoing feedback, guidance, and correction
- Grounding students' learning in real-world contexts and applications so that students connect new information to their lives outside of school
- Providing social contexts and peer modeling for learning
- Accurately assessing student progress and modifying instruction accordingly

A consistent instructional cycle that incorporates all of these elements enables students to organize, store, and retrieve new knowledge, while strengthening interconnections between the pieces of information in their mental "maps" so that the information will be available to them for recall, transfer, and future use. When students have opportunity to practice skills to the point of automaticity their working memory is freed for new tasks and they are able to see patterns, relationships, and discrepancies in problems that they would have missed without such practice (Anderson, Greeno, Reder, & Simon, 2000; Bransford, Brown, & Cocking, 2000; Collins, Brown, & Newman, 1989; Ellis & Worthington, 1994; Good & Brophy, 2003; Marzano, Gaddy, & Dean, 2000; Means & Knapp, 1991; Pressley, et al, 1995; Rosenshine, 2002; Rosenshine & Meister, 1995; Stevenson & Stigler, 1992; Wenglinsky, 2002, 2004).

B. What the Research Says About Vocabulary Instruction in Mathematics

Language plays a significant role in mathematics. Therefore, direct instruction of key vocabulary is a critical element in raising student achievement in mathematics. Striving readers, English language learners, and students who have language or developmental challenges all require additional support in developing academic vocabulary. Because students approach a lesson or problem with much, little, or incorrect prior knowledge of the topic or terminology at hand, effective teachers use questions, cues, and advance organizers to discern what and how much their students already know, and whether they have misconceptions (Marzano, Gaddy, & Dean, 2000).

In class discussions, students can externalize and discuss thought processes that they may not have consciously considered if they were working alone and become familiar and adept at communicating in the "mathematical register" – the specialized vocabulary of mathematics. By describing problem solving processes, students can practice vocabulary, syntax, semantics, and discourse features related specifically to learning mathematics. To help students gain a deep understanding of abstract concepts, a variety of approaches and strategies have been proven useful for explicitly teaching word meanings. Research-confirmed methods for vocabulary instruction include:

- Using students' sociocultural and linguistic experiences to make mathematical connections between natural language and mathematics-specific language
- Presenting students with explanations and definitions of target words
- Using objects
- Providing demonstrations
- Using facial expressions, gestures, and dramatizations
- Using graphic organizers
- Asking students to determine definitions from context
- Asking students to produce their own definitions and then giving them feedback
- Asking students to generate nonlinguistic representations of new terms or phrases
- Asking students to compare and contrast new information with other knowledge and processes, identifying similarities and differences
- Asking students to create their own metaphors and analogies
- Clarifying and elaborating on key concepts and vocabulary by explaining in the student's native language
- Presenting fewer than seven new words at a time and having students work on these over the course of several lessons so that they learn the meanings at a deep level of understanding
- Asking students to write and use the word in a variety of contexts
- Helping students link the words to relevant, familiar experiences in their own lives
- Writing key terms or phrases on the board, providing students a resource to use in their own speech.
- Using visual, kinesthetic, and auditory teaching approaches to explicitly move students from concrete to abstract understanding and performance and to give English learners a variety of ways to connect with the information being presented
- Adjusting teacher speech to ensure student understanding – using controlled vocabulary, facing students, speaking slowly, enunciating clearly, pausing frequently, and paraphrasing or repeating difficult concepts
- Asking students to provide reasons for their answers and explanations for their solutions
- Focusing on student meaning, not grammar
- Accepting and building on student responses – "revoicing" student statements using more technical terms in order to give students more linguistic input and more time to process complex material
- Modeling academic language
- Using students' own terminology if it seems to capture meaning in a way that will be understood by other students

- Encouraging students to express their ideas by responding with phrases like "tell me more about that" or "why do you think so?"
- Using visuals, manipulatives, and concrete materials
- Using hands-on learning activities that involve academic language
- Checking frequently for understanding by eliciting requests for clarification and posing questions
- Rewriting word problems in simpler terms

(Brenner, 1998; Furner, Yahya, & Duffy, 2005; Gersten & Baker, 2000; Jarrett, 1999; Khisty & Chval, 2002; Laturnau, 2001; Marzano, 1998; Marzano, Gaddy, & Dean, 2000; Moschkovich, 1999; Reed & Railsback, 2003; Short & Echevarria, 2004/2005)

C. What the Research Says about Multiple Representations of Mathematical Concepts

When students "see" or experience mathematical ideas through words, pictures, or concrete objects that represent the ideas in linguistic and nonlinguistic ways, they learn to translate between and among these multiple representations, resulting in deeper understanding and improved performance.

Students typically move through three stages, from the simple to the complex, as they develop understanding of a mathematical concept (Bruner, 1966):

- The *enactive* stage: Manipulating concrete materials
- The *iconic* stage: Working with pictures, graphs, diagrams, and charts
- The *symbolic* stage: Expressing mathematical ideas through numerals, formulas, and theorems

Further, mathematical understanding depends upon the quality of the connections students are able to build between:

- Formal and informal mathematical experience
- New information and prior knowledge
- Conceptual understanding and procedural skills

However, students do not automatically make these connections or transfer their informal or concrete mathematical understandings to formal, symbolic mathematics. They need to explicitly discuss these connections, argue why solutions are reasonable or unreasonable, and explain how they know what they know (Brenner et al, 1997; Hiebert & Carpenter, 1992; Lampert, 1986; Yetkin, 2003).

Therefore, students benefit from exploring new concepts through an interactive process with teachers and other students that includes creating non-standard representations which they can then connect to standard forms. The ability to represent mathematical ideas in a variety of forms is especially vital to conceptual understanding, strategic competence, adaptive reasoning, and problem solving. Thus, representations serve both as teaching tools and as the means by which students can think, explain, determine, and justify mathematical solutions (Boerst, 2005;

Cifarelli, 1998; Cobb, Yackel, & Wood, 1992; Goldin, 2002; Kilpatrick, Swafford, & Findell, 2001; Marzano, Gaddy, & Dean, 2000; Pape & Tchoshanov, 2001).

D. What the Research Says About Mathematics Instruction for Students with Special Needs

In addition to language needs, any classroom may include students with a variety of other learning challenges to which instruction must be adapted. Students who struggle to learn mathematics may have learning challenges in one or more areas. For example:

Students with *attention* challenges may have difficulty

- Maintaining attention to steps in algorithms or problem solving
- Sustaining attention to critical instruction (e.g., teacher modeling)

Students with v*isual-spatial* problems may have difficulty

- Maintaining their place on worksheets
- Differentiating between numbers (e.g., 6 and 9; 2 and 5; or 17 and 71), coins, the operation symbols, and clock hands
- Writing across the paper in a straight line
- Relating to directional aspects of math, for example, in problems involving up-down (e.g., addition), left-right (regrouping), and aligning of numbers
- Using a number line

Students with *auditory-processing* difficulties may have difficulty

- Responding to oral drills
- Counting on from within a sequence

Students with *memory* challenges may have difficulty

- Retaining math facts or new information
- Remembering steps in an algorithm
- Performing proficiently on review lessons or mixed probes
- Telling time
- Solving multi-step word problems

Students with *motor function* issues may have difficulty

- Writing numbers legibly in small spaces
- Writing numbers quickly and accurately

Students with *cognitive and metacognitive* challenges may have difficulty

- Assessing their abilities to solve problems
- Identifying and selecting appropriate strategies
- Organizing information
- Monitoring problem-solving processes
- Evaluating problems for accuracy
- Generalizing strategies to appropriate situations

(Miller & Mercer, 1997)

Students with learning challenges may be discouraged and disinclined to try when it comes to improving their skills in mathematics. Research-validated strategies for re-orienting their attitudes and ensuring success include:

- Moving from concrete to abstract
- Including physical and pictorial models (e.g., manipulatives and diagrams)
- Involving students in setting challenging but attainable learning goals for themselves
- Modeling enthusiasm toward mathematics
- Maintaining a lively instructional pace
- Using progress charts for feedback on how well students are progressing relative to their own record
- Communicating positive expectations for student learning
- Reinforcing student effort
- Using auditory and kinesthetic approaches (e.g., rhymes, raps, and chants) to help students remember concepts
- Practicing step-by-step processes for most tasks
- Using think-aloud techniques when modeling steps to solve problems
- Asking students to verbalize their thinking as they solve problems
- Discussing the relevance of math skills to real-life problems

(Mastriopieri et al, 1991; Miller, Butler, & Lee, 1998; Miller & Mercer, 1992, 1997; Witzel, Smith, & Brownell, 2001)

Of particular note, the skillful use of concrete instructional materials (manipulatives) and "hands-on" approaches have been found to improve achievement and attitudes toward mathematics among all types of students, including those with special needs. Such materials and activities aid student understanding of concepts and processes, increase cognitive flexibility, provide tools for problem solving, and reduce student anxiety. Further, active, physical experiences with mathematical concepts allow students to see how principles are derived before they are discussed in abstract terms or formalized (Marzano, Gaddy, & Dean, 2000; Raphael & Wahlstrom, 1989; Sowell, 1989; Stevenson & Stigler, 1992; Suydam, 1986; Wenglinsky, 2002, 2004).

Manipulatives are most effective in helping students learn basic computational processes, place value, and geometric concepts. When students have constructed concepts using concrete materials, they retain and draw upon those concepts later through mental imagery when the materials are not present. Concrete manipulatives are most effectively used in initial instruction about concepts and processes – once students have learned rote procedures and algorithms, manipulatives are less helpful (Fuson, 1992; Fuson & Briars, 1990; Sowell, 1989; Thompson, 1992).

Research also shows that when using concrete materials to illustrate mathematics concepts, it is important that teachers not assume that students will automatically make the desired connections between concrete representations and abstract mathematical ideas. Interpreting or translating the meaning of the concrete example may require very complex cognitive processing. Teachers need to intervene frequently during the instruction process to check student understanding, focus on the underlying mathematical ideas, and explicitly help students move from work with concrete manipulatives to corresponding work with mathematical symbols (Ball, 1992; Fuson, 1992; Hiebert & Wearne, 1988; Johnson, 2000)

E. What the Research Says About Assessment in Mathematics Instruction

An effective mathematics program includes three types of assessment (McTighe & O'Connor, 2005):

- Broad **diagnostic assessment** to determine students' entry-level knowledge and skills for purposes of appropriate placement within the program
- Ongoing **formative assessment** – daily and weekly monitoring of student progress toward achieving the standards
- **Summative evaluation** at the end of each unit or course to provide specific and detailed information about which learning standards have or have not been achieved.

Of these three vital assessment types, ongoing formative assessment is particularly critical for helping teachers make the most efficient use of time to advance student learning. Informal daily progress monitoring and frequent, well-aligned, brief formal assessments give teachers information about students' conceptual understanding, procedural fluency, and problem-solving ability that they can use to guide further instructional planning.

Research shows that ongoing formative assessment develops students' capacity to become reflective, self-managing learners. Regular monitoring of student learning provides students with constructive feedback about their progress toward achieving the standards and guides them as to how to improve. Therefore, students who receive focused, helpful comments about their performance on assessment tasks engage more productively in their work (Black et al.,2003, 2004; Black & Wiliam, 1998; Bransford, Brown, & Cocking, 2000; Marzano, Gaddy, & Dean, 2002; Shepard, 2005).

F. What the Research Says About What Teachers of Mathematics Need

In order to teach effectively, not only must teachers understand mathematics in a deep and flexible way; they must also understand how students learn mathematics. Mathematical knowledge needed for teaching, known as *pedagogical content knowledge*, is a more complex phenomenon than can be captured in measures of courses taken or degrees earned.

<u>What Teachers of Mathematics Need to Know and Be Able to Do:</u> When teachers' knowledge of mathematics or their knowledge of teaching mathematics is limited, they may at best fall short of providing their students with powerful mathematical experiences. At worst, they may actually misinform and mislead students because of their own misconceptions or because they tend to interpret a student's explanation or question in light of their own mathematical understanding, misjudging what the student is actually thinking (Mewborn, 2003; Prawat et al, 1992; Thompson & Thompson, 1994, 1996).

Pedagogical content knowledge is "the blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of learners for instruction" (Shulman, 1987, p.8). Because teaching is situation-specific, teachers must continuously adapt and adjust their practices in an effort to help every student learn:

> … teachers of mathematics not only need to calculate correctly but also need to know how to use pictures or diagrams to represent mathematics concepts and procedures to students, provide students with explanations for common rules and mathematical procedures, …analyze students' solutions and explanations….and provide students with examples of mathematical concepts, algorithms, or proofs. (Hill, Rowan, & Ball, 2005, p. 373)

Teachers need to understand how to use a variety of assessments to regularly monitor student learning and how to adjust instruction according to assessment data, because:

> …students' ability depends partly on how well teachers probe, understand, and use their work. Even the strengths or disadvantages that students are said to "bring" to instruction are partly a matter of what their teachers can see and hear in students' work and how skillfully they recognize and respond to them. Students' ability is in part interactively determined.… "instructional capacity" is not a fixed attribute of teachers or students or materials, but a variable feature of interaction among them. (Cohen, Raudenbush, & Ball, 2000, p. 13)

Teachers need to understand what makes the learning of specific topics difficult or easy for students and know how to provide clearer explanations, make efficient use of class time, and engage students in inquiry by using whole-class pedagogical techniques. They should be able to provide counterexamples to expose errors in students' thinking, follow through on students' comments to lead to a contradiction or a viable solution, apply a student's method to a simpler or related problem, understand a student's alternative method, and incorporate a student's alternative method into instruction (Fernandez, 1997; Hill, Rowan, & Ball, 2005; Ma, 1999; Stigler & Hiebert, 1999; Wenglinsky, 2002).

Effective Professional Development Support for Teachers of Mathematics: Teachers benefit from opportunities to learn mathematics in the ways in which they are expected to teach it to their students. Research shows that well designed curriculum materials can shape teachers' ideas about their practice, support and improve teachers' work to help students learn mathematics, and contribute to teachers' mathematical understanding (Cohen, Raudenbush, & Ball, 2000).

Research has also clearly demonstrated that sustained professional development activities that are embedded in teachers' day-to-day work lives are essential to help teachers develop the depth of understanding they must have of mathematics content and of how to best to help their students learn it. Teachers need opportunities to wrestle with important mathematical ideas, justify their thinking to peers, and investigate alternative solutions proposed by others. They need to share student work, observe and obtain feedback from colleagues, and reconsider their conceptions of what it means to do mathematics in a context that allows them to try what they learn in their classrooms (Kilpatrick, Swafford, & Findell, 2001; Mewborn, 2003; Schifter, 1998).

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